ON BERGMAN COMPLETENESS OF NON-HYPERCONVEX DOMAINS

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Abstract. In the paper we study the problems of the boundary behaviour of the Bergman kernel and the Bergman completeness in some classes of bounded pseudoconvex domains, which contain also non-hyperconvex domains. Among the classes for which we prove the Bergman completeness and the convergence of the Bergman kernel to infinity while tending to the boundary are all bounded pseudonvex balanced domains, all bounded Hartogs domains with balanced fibers over regular domains and some bounded Laurent-Hartogs domains.

0. Introduction. The aim of the paper is to present some new results concerning Bergman completeness and the boundary behaviour of the Bergman kernel in bounded pseudoconvex but not necessarily hyperconvex domains. We are interested in the following exhausting property of the Bergman kernel:

$$K_D(z) \to \infty \text{ as } z \to \partial D. \tag{$*$}$$

The starting point for our considerations may be the following two recent results:

– any bounded hyperconvex domain satisfies $(*)$ (see [Ohs 2]),

– any bounded hyperconvex domain is Bergman complete (see [Blo-Pfl] and [Her]).

Both properties mentioned above are closely related. In particular, the Bergman completeness is often proved after proving the property $(*)$. To the best of our knowledge there are no known examples of bounded Bergman complete domains not satisfying $(*)$.

The existence of non-hyperconvex bounded domains with $(*)$ is very well-known and easy (take the Hartogs triangle). On the other hand the existence of bounded pseudoconvex but non-hyperconvex Bergman complete domains is not so trivial but also known (see [Chen], [Her], [Zwo]).

In our paper we shall present a class of domains satisfying the above properties. The classes of domains which we consider are the following: bounded pseudoconvex balanced domains, Hartogs domains with m-dimensional balanced fibers, Hartogs-Laurent domains and Zalcman type domains (domains in the unit disc with complements composing of infinitely many closed discs).

Among others we prove the following results.

All bounded pseudoconvex balanced domains satisfy $(*)$ and are Bergman complete. The latter result gives the positive answer to the question posed in [Jar-Pfl 1]

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and [Jar-Pfl 2]. Note that if the Minkowski functional of the considered domain is continuous then the domain is hyperconvex and the result follows from the above mentioned theorems.

Any bounded pseudoconvex Hartogs domain with m-dimensional balanced fibers over a domain with the property $(*)$ satisfies $(*)$. Any bounded pseudoconvex Hartogs domain over a c^i-complete domain (which implies automatically Bergman completeness) is Bergman complete. In particular, there are bounded and pseudoconvex non-fat domains that are Bergman complete and satisfy $(*)$.

On the other hand we show that there are bounded fat domains in $\mathbb{C}$ (some Zalcman type domains) not satisfying $(*)$ – this gives an answer to a question posed in [Jar-Pfl 2].

1. Definitions and known results. Let us denote by E the unit disc in $\mathbb{C}$.

Let D be a bounded domain in $\mathbb{C}^n$. Let us denote by $L_h^2(D)$ square integrable holomorphic functions on D. $L_h^2(D)$ is a Hilbert space with the scalar product induced from $L^2(D)$. Let us define the *Bergman kernel of D*

$$K_D(z)=\sup\{\frac{|f(z)|^2}{||f||^2_{L^2(D)}}:f\in L_h^2(D), f\not\equiv 0\}.$$

Among other well-known properties let us recall only two of them (see e.g [Jar-Pfl 2]).

If $D_1\subset D_2$ are bounded domains in $\mathbb{C}^n$ then $K_{D_2}(z)\le K_{D_1}(z)$, $z\in D_1$.

If $\{D_j\}_{j=1}^{\infty}$ is an increasing sequence of domains in $\mathbb{C}^n$ whose union is a bounded domain D, then K_{D_j} tends decreasingly and locally uniformly to K_D.

It is well-known that $\log K_D$ is a smooth plurisubharmonic function. Therefore, we may define

$$\beta_D(z;X):=\left(\sum_{j,k=1}^n\frac{\partial^2\log K_D(z)}{\partial z_j\partial\bar z_k}X_j\bar X_k\right)^{1/2},\ z\in D,\ X\in\mathbb{C}^n.$$

β_D is a pseudometric called *the Bergman pseudometric.*

For $w,z\in D$ we put

$$b_D(w,z):=\inf\{L_{\beta_D}(\alpha)\},$$

where the infimum is taken over piecewise C^1-curves $\alpha:[0,1]\mapsto D$ joining w and z and $L_{\beta_D}(\alpha):=\int_0^1\beta_D(\alpha(t);\alpha'(t))dt$.

We call b_D *the Bergman distance of D.*

The Bergman distance (as well as the Bergman metric) is invariant with respect to biholomorphic mappings. In other words, for any biholomorphic mapping $F: D\mapsto G$ $(D,G\subset\subset\mathbb{C}^n)$ we have

$$b_G(F(w),F(z))=b_D(w,z),\quad \beta_G(F(w);F'(w)X)=\beta_D(w;X),\ w,z\in D, X\in\mathbb{C}^n.$$

A bounded domain D is called *Bergman complete* if any b_D-Cauchy sequence is convergent to some point in D with respect to the standard topology of D.

Any bounded Bergman complete domain is pseudoconvex (see [Bre]). Let us recall that a bounded domain D is called *hyperconvex* if it admits a continuous negative plurisubharmonic exhaustion function. Now we may formulate the following very general result:

Theorem 1.1 (see [Bło-Pfl], [Her], [Ohs 2])**.** *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Then D satisfies $(*)$ and D is Bergman complete.*

Our aim is to study the boundary behaviour of the Bergman kernel and the problem of Bergman completeness. We shall make use of following powerful tools; namely, the extension theorem of L_h^2-functions, localization principle of the Bergman kernel and the Bergman metric and criteria for a domain to be Bergman complete and for the Bergman kernel to tend to infinity near the fixed point from the boundary. Let us recall below these results.

Theorem 1.2 (see [Ohs-Tak])**.** *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let H be any affine subspace of $\mathbb{C}^n$. Then there is a constant $C \in \mathbb{R}$ dependent only on diameter of D such that for any $f \in L_h^2(D \cap H)$ there is $F \in L_h^2(D)$ such that $||F||_{L^2(D)} \leq C||f||_{L^2(D\cap H)}$.*

In particular, we get from Theorem 1.2 that

$$K_{D\cap H}(z) \leq \tilde{C} K_D(z),\ z \in D \cap H, \tag{1.1}$$

where $\tilde{C} \in \mathbb{R}$ is a constant dependent only on diameter of D.

Theorem 1.3 (see [Die-For-Her], [Ohs 1])**.** *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $z^0 \in \partial D$. Then for any neighbourhoods $U_1 = U_1(z^0) \subset\subset U_2(z^0) = U_2$ there is a positive constant C such that for any connected component V of $D \cap U_2$ and for any $z \in U_1 \cap V$, $X \in \mathbb{C}^n$ we have:*

$$\frac{1}{C} K_V(z) \leq K_D(z) \leq K_V(z),$$
$$\frac{1}{C} \beta_V(z; X) \leq \beta_D(z; X) \leq \beta_V(z; X).$$

Theorem 1.4 (see [Kob], [Pfl 1])**.** *Let D be a bounded domain such that $(*)$ is satisfied and $H^\infty(D)$ is dense in $L_h^2(D)$. Then D is Bergman complete.*

Theorem 1.5 (see [Pfl 1])**.** *Let D be a bounded pseudoconvex domain. Let $z^0 \in \partial D$ be such that there are $r \in (0,1]$, $\varepsilon \geq 1$ and a sequence $\{z^\nu\}_{\nu=1}^\infty$ of points from $\mathbb{C}^n \setminus \bar{D}$ tending to z^0 such that $B(z, r||z^\nu - z^0||^\varepsilon) \cap D = \emptyset$ (so called 'outer cone condition'). Then $\lim_{z\to z^0} K_D(z) = \infty$.*

2. Balanced domains. Recall that a set D is *balanced* if $z \in D$ and $\lambda \in \bar{E}$ implies $\lambda z \in D$.

In this section we deal with bounded pseudoconvex balanced domains. We prove the following result.

Theorem 2.1. *Let D be a bounded pseudoconvex balanced domain. Then D satisfies $(*)$ and D is Bergman complete.*

Note that if the Minkowski functional of D is continuous then D is hyperconvex and the result follows from Theorem 1.1. Additionally, in this case the theorem has already been known for a long time (see [Jar-Pfl 1]). Using only a little more refined methods than the ones used in the last paper we prove the theorem in general case. Let us mention here that the problem whether bounded pseudoconvex balanced domains are Bergman complete has been stated in [Jar-Pfl 1] and [Jar-Pfl 2].

Proof of Theorem 2.1. First we prove the property $(*)$. Take any point $z^0 \in \partial D$. Fix any $M \in \mathbb{R}$. In view of (1.1) (applied to $H = \mathbb{C}z^0$ – remember that $\mathbb{C}z^0 \cap D$ is a disc) there is some $z^1 = sz^0$, $0 < s < 1$ such that $z^1 \in D$ and $K_D(z^1) > M$. It follows from continuity of K_D that there is some open neighbourhood $U \subset D$ of z^1 such that $K_D(z) > M$ for $z \in U$. Note that for any $z \in U$ the function

$$u_z : \frac{1}{h(z)} E \ni \lambda \mapsto K_D(\lambda z)$$

is subharmonic and radial. Therefore, $u_z(t)$, $0 \le t < \frac{1}{h(z)}$ is increasing (see e.g. [Jak-Jar]). Consequently, $K_D(z) > M$ for any $z \in ([1,\infty)U) \cap D$. Since $[1,\infty)U$ is a neighbourhood of z^0, we finish the proof.

To finish the proof it is sufficient to show that $H^\infty(D)$ is dense in $L^2_h(D)$ (and then use Theorem 1.4).

It is well-known that any holomorphic function F on D is a local uniform limit of a series $\sum_{k=0}^\infty Q_k(z)$, where Q_k is a homogeneous polynomial of degree k (see e.g. [Jak-Jar]). Since all Q_k are orthogonal (in $L^2_h(D)$) and there is an exhausting family of compact balanced sets of the domain D (on each of them the functions Q_k are orthogonal), the standard approximation process leads to the convergence of $F_N := \sum_{k=0}^N Q_k$ to F in $L^2(D)$ norm (under the assumption that $F \in L^2_h(D)$). Since D is bounded, all F_N's are bounded, which finishes the proof. $\square$

3. Hartogs domains. In the present section we consider bounded pseudoconvex Hartogs domains with m-dimensional balanced fibers. Let $G_D \subset \mathbb{C}^{n+m}$ denote a bounded pseudoconvex Hartogs domain over $D \subset \mathbb{C}^n$ with m-dimensional balanced fibers, i.e.

$$G_D = \{(z,w) \in D \times \mathbb{C}^m : H(z,w) < 1\},$$

where D is bounded and pseudoconvex, $\log H$ is plurisubharmonic on $D \times \mathbb{C}^m$, $H(z,\lambda w) = |\lambda| H(z,w)$, $(z,w) \in D \times \mathbb{C}^m$, $\lambda \in \mathbb{C}$, and G_D is bounded (i.e. $H(z,w) \ge C||w||$ for some $C > 0$, $z \in D$, $w \in \mathbb{C}^m$).

Let G_D be as above. For any $f \in L^2_h(D)$ we define a function $F(z,w) := f(z)$, $(z,w) \in G_D$. Since $G_D \subset D \times (RE)^m$ for some $R > 0$, we easily get that $||F||_{L^2_h(G_D)} \le C_1 ||f||_{L^2_h(D)}$ for some $C_1 > 0$ independent of the choice of f (therefore, $F \in L^2_h(G_D)$). In particular, we get that

$$K_D(z) \le C_2 K_{G_D}(z,0),\ z \in D. \tag{3.1}$$

Theorem 3.1. *Let G_D be a bounded pseudoconvex Hartogs domain over D with m-dimensional balanced fibers. Fix a point $(z^0,w^0) \in \partial G_D$. Assume that one of the following three conditions is satisfied:*

$$z^0 \in D, \tag{i}$$

$$z^0 \in \partial D \text{ and } K_D(z) \to \infty \text{ as } z \to z^0, \tag{ii}$$

$$\text{there is some neighbourhood } U \text{ of } (z^0,w^0) \text{ such that} \tag{iii}$$

$U \cap G_D \subset \{(z,w) \in \mathbb{C}^{n+m} : ||w|| < ||z - z^0||^\delta\}$ for some $\delta > 0$ (in particular, $w^0 = 0$).

Then $K_{G_D}(z,w) \to \infty$ as $(z,w) \to (z^0,w^0)$.

In particular, if D satisfies $(*)$, then G_D satisfies $(*)$.

Proof. Consider the case $z^0 \in D$ (that is we consider the case (i)). Then $H(z^0, w^0) \geq 1$. In view of the L_h^2-extension theorem for any $M \in \mathbb{R}$ there is $w^1 = tw^0$, $0 < t < 1$, such that $(z^0, w^1) \in G_D$ and $K_{G_D}(z^0, w^1) > M$. Continuity of K_{G_D} gives us the existence of an open neighbourhood $U := U_1 \times U_2$ of (z^0, w^1) in G_D (with $0 \notin U_2$) such that $K_{G_D}(z, w) > M$ for $(z, w) \in U$. Similarly, as earlier considering the function

$$u_{(z,w)} : \frac{1}{H(z,w)}E \ni \lambda \mapsto K_{G_D}(z, \lambda w)$$

we get a radial subharmonic function such that $u_{(z,w)}(1) > M$, which gives us that $K_{G_D}(z, w) > M$ for $(z, w) \in (U_1 \times [1, \infty)U_2) \cap G_D$.

Consider now the case (ii). Then $z^0 \in \partial D$. It follows from (3.1) and (ii) that for any $M \in \mathbb{R}$ there is some open neighbourhood U of z^0 such that $K_{G_D}(z, 0) > M$, $z \in U \cap D$.

Fix $z \in U \cap D$. Fix additionally for a while w such that $0 < H(z, w) < 1$. Then the function $\frac{1}{H(z,w)}E \ni \lambda \mapsto K_{G_D}(z, \lambda w)$ is larger than M at 0 and is radial and subharmonic; therefore, increasing. Consequently, $K_{G_D}(z, w) > M$ for any w with $H(z, w) < 1$. Since $z \in U$ was chosen arbitrarily we get that $K_{G_D}(z, w) > M$ for any $(z, w) \in G_D$ with $z \in U$.

We are left with the case (iii). Without loss of generality we may assume that $z^0 = 0$. Consider points $(0, w_0) \notin \bar{D}$ (i.e. $w_0 \neq 0$). Let us consider the balls $B((0, w_0), r||w_0||^\varepsilon)$, where $\varepsilon > 0$, $0 < r < 1$ will be chosen later (independently of w_0). Our aim is to verify that the outer cone condition form Theorem 1.5 is satisfied for a suitable $0 < r \leq 1$ and $\varepsilon \geq 1$.

Fix $r = \frac{1}{2}$. Consider only $||w_0|| < \frac{1}{2}$. Take a point $(z, w) \in B((0, w_0), \frac{1}{2}||w_0||^\varepsilon) \cap \bar{G}_D$. Then $||z|| < \frac{1}{2}||w_0||^\varepsilon$ and $||w_0|| - ||w|| \leq ||w - w_0|| \leq \frac{1}{2}||w_0||^\varepsilon$. Consequently, $||w_0|| - \frac{1}{2}||w_0||^\varepsilon \leq ||w|| \leq ||z||^\delta < (\frac{1}{2})^\delta ||w_0||^{\delta\varepsilon}$. So assuming that ε is large enough ($\varepsilon - 1 > 0$, $\varepsilon\delta - 1 > 0$, $\delta + \varepsilon\delta - 2 > 0$) we get:

$$\frac{1}{2} < 1 - \frac{1}{2}||w_0||^{\varepsilon-1} < (\frac{1}{2})^\delta ||w_0||^{\varepsilon\delta-1} < \frac{1}{2}$$

– contradiction. Therefore, in view of Theorem 1.5 we finish the proof. □

The idea of the condition (iii) comes from generalizing the phenomenon, which appears in the Hartogs triangle and the point $(z^0, w^0) = (0, 0)$.

It turns out that there are bounded pseudoconvex Hartogs domains and points from the boundary, which do not satisfy any from the conditions (i)–(iii) but such that the limes as in Theorem 3.1 exists.

Example 3.2. Let $\{a_j\}_{j=1}^\infty \subset (0, 1)$ be a sequence tending to 0. Let us define $u_k(\lambda) = \log\left(\sum_{j=1}^k (\frac{a_j}{2|\lambda - a_j|})^{n_j}\right)$, $\lambda \in E \setminus \{a_1, \dots, a_k\}$, where $n_j \geq j$. Note that $u_k(0) < 0$, $k = 1, 2, \dots$. Define $u := \lim_{k\to\infty} u_k = \log\left(\sum_{j=1}^\infty (\frac{a_j}{2|\lambda - a_j|})^{n_j}\right)$ on $E_\infty := E \setminus (\{a_j\}_{j=1}^\infty \cup \{0\})$. The construction ensures us that the sequence $\{u_k\}_{k=1}^\infty$ is locally bounded from above and globally bounded from below increasing sequence on E_∞ and, therefore, u is a subharmonic function on E_∞ bounded from below. Moreover, $\lim_{x<0, x\to 0} u(x) \leq 0$. Define $G_{E_\infty} := \{(z, w) \in E_\infty \times \mathbb{C} : |w| < \exp(-u(z))\}$. Then

that the point $(0,0)$ does not satisfy any from the conditions (i)–(iii) but one may easily verify that, choosing if necessary n_j larger, the outer cone condition from Theorem 1.5 is satisfied (for instance for points (a_j, a_j)). Therefore, the claim of Theorem 3.1 is also satisfied. Note that $\{(0,0)\} \subsetneq \partial G_{E_\infty} \cap (\{0\} \times \mathbb{C}^m)$.

We may prove even more. Namely, the domain G_{E_∞} satisfies $(*)$. In fact, the points $(z,w) \in \partial G_{E_\infty}$, $z \in \partial E$, satisfy (ii). The points $(a_k, w) \in \partial G_{E_\infty}$ (and then automatically $w = 0$) satisfy (iii). The points $(z,w) \in \partial G_{E_\infty}$, $z \in E_\infty$, satisfy (i). Finally, one may easily verify (proceeding similarly as in the case $(0,0)$) that the points $(0,w) \in \partial G_{E_\infty}$ satisfy the outer cone condition from Theorem 1.5.

Lemma 3.3. *Let G_D be a bounded pseudoconvex Hartogs domain over D with m-dimensional balanced fibers such that $H^\infty(D)$ is dense in $L^2_h(D)$ and, additionally, assume that there is some $\varepsilon > 0$ such that $D \times P(0,\varepsilon) \subset G_D$. Then $H^\infty(G_D)$ is dense in $L^2_h(G_D)$.*

Proof. Take $F \in L^2_h(G_D)$. We know that

$$F(z,w) = \sum_{\nu=0}^{\infty} F_\nu(z,w) := \sum_{\nu=0}^{\infty} \sum_{\beta \in \mathbb{Z}_+^m : |\beta| = \nu} f_\beta(z) w^\beta$$

where the convergence of $G_N := \sum_{\nu=0}^{N} F_\nu$ to F is locally uniform (see e.g. [Jak-Jar]). Consequently, because of orthogonality of w^β, similarly as in the proof of Theorem 2.1 the functions G_N converge in $L^2(G_D)$ to F. It is therefore sufficient to approximate $f_\beta(z) w^\beta$ with bounded functions. But because of the assumption of the lemma one may easily conclude from the Fubini theorem that $f_\beta \in L^2_h(D)$ so $h_N(z) w^\beta$, where $h_N \in H^\infty(D)$ and $h_N \to f_\beta$ in $L^2_h(D)$, tends to $f_\beta(z) w^\beta$ in $L^2_h(G_D)$. $\square$

Remark 3.4. Note that the assumption $D \times P(0,\varepsilon) \subset G_D$ is essential. For instance, $H^\infty(E_*) = H^\infty(E)|_{E_*}$ is dense in $L^2_h(E_*) = L^2_h(E)|_{E_*}$ and $H^\infty(G_{E_*})$ is not dense in $L^2_h(G_{E_*})$, where G_{E_*} is the Hartogs triangle, $G_{E_*} := \{(z,w) \in E_* \times \mathbb{C} : |w| < |z|\}$.

Theorem 3.5. *Let G_D be a bounded pseudoconvex Hartogs domain over D with m-dimensional balanced fibers. Assume that D satisfies $(*)$, $H^\infty(D)$ is dense in $L^2_h(D)$ and there is $\varepsilon > 0$ such that $D \times P(0,\varepsilon) \subset G_D$. Then G_D is Bergman complete.*

Proof. Combine Theorem 3.1, Lemma 3.3 and Theorem 1.4. $\square$

Note that Theorem 3.5 cannot be even applied to arbitrary pseudoconvex bounded Hartogs domain with one dimensional fibers. However, small change in assumptions on the domain D in Theorem 3.5 will make it possible to prove Bergman completeness of G_D without additional assumptions on the shape of G_D. But before formulating the result we have to introduce the notion of the inner Carathéodory pseudodistance.

For a domain $D \subset \mathbb{C}^n$ we define *the Carathéodory-Reiffen pseudometric*

$$\gamma_D(z;X) := \sup\{|f'(z)X| : \ f \in \mathcal{O}(D,E), \ f(z) = 0\}, \ z \in D, \ X \in \mathbb{C}^n.$$

The inner Carathéodory pseudodistance is the integrated form of γ_D, i.e.

$$c^i_D(w,z) := \inf\{L_{\gamma_D}(\alpha) : \alpha : [0,1] \to D \text{ is a piecewise } C^1 \text{ curve joining } w \text{ and } z\}$$

where $L_{\gamma_D}(\alpha) := \int_0^1 \gamma_D(\alpha(t); \alpha'(t))dt$. It is well-known that holomorphic mappings are contractions with respect to c^i (i.e. $c^i_G(F(w), F(z)) \leq c^i_D(w, z)$ for any $F \in \mathcal{O}(D, G)$, $w, z \in D$). The last property is not shared by the Bergman distance (in the class of bounded domains – see e.g. [Jar-Pfl 2]). We have additionally that (see e.g. [Jar-Pfl 2])

$$c^i_D \leq b_D. \tag{3.2}$$

Exactly as in the case of the Bergman distance we introduce for bounded domains the notion of c^i-completeness.

Theorem 3.6. *Let G_D be a bounded pseudoconvex Hartogs domain over D with m-dimensional balanced fibers. Assume that D is c^i-complete. Then G_D is Bergman complete.*

Proof. Take any point $(z_0, w_0) \in \partial G_D$. Suppose that there is a b_{G_D}-Cauchy sequence $\{(z_\nu, w_\nu)\}$ tending (in the natural topology of D) to (z_0, w_0). Because of (3.2) and the contractivity of c^i_D with respect to the projection we exclude the case $z^0 \in \partial D$.

So assume that $z^0 \in D$. Let U_1, U_2 be small open balls with the centre at z^0 such that $U_1 \subset\subset U_2 \subset\subset D$. There is a sequence of C^1-piecewise curves $\gamma_{\nu,\mu} : [0, 1] \mapsto G_D$ such that $\gamma_{\nu,\mu}(0) = (z_\nu, w_\nu)$, $\gamma_{\nu,\mu}(1) = (z_\mu, w_\mu)$ and $L_{\beta_{G_D}}(\gamma_{\nu,\mu}) < b_{G_D}((z_\nu, w_\nu), (z_\mu, w_\mu)) + \frac{1}{\nu}$, $1 \leq \nu < \mu$. We claim that there is some ν_0 such that $\gamma_{\nu,\mu}([0, 1]) \subset G_{U_1}$ ($G_{U_j} := (U_j \times \mathbb{C}^m) \cap G_D$, $j = 1, 2$) for $\mu > \nu > \nu_0$. Actually, if it were not the case, then there would be a sequence of $t_k \in (0, 1)$ such that $(u_k, v_k) := \gamma_{\nu_k,\mu_k}(t_k) \notin G_{U_1}$ (so $u_k \notin U_1$) and $b_{G_D}((z_{\nu_k}, w_{\nu_k}), (u_k, v_k)) \to 0$ as k tends to infinity. But then also

$$0 \leq c^i_D(z_{n_k}, u_k) \leq c^i_{G_D}((z_{n_k}, w_{n_k}), (u_k, v_k)) \to 0$$

– contradiction (to the boundedness of D).

Note that G_{U_j} satisfies the assumptions of Theorem 3.5, so G_{U_j} is Bergman complete, $j = 1, 2$.

Applying the localization principle of the Bergman metric (Theorem 1.3) we get that

$$b_{G_{U_2}}((z_\nu, w_\nu), (z_\mu, w_\mu)) \leq L_{\beta_{G_{U_2}}}(\gamma_{\nu,\mu}) \leq$$
$$CL_{\beta_{G_D}}(\gamma_{\nu,\mu}) \leq C(b_{G_D}((z_\nu, w_\nu), (z_\mu, w_\mu)) + \frac{1}{\nu}),\ \mu > \nu > \nu_0$$

so $\{(z_\nu, w_\nu)\}_{\nu>>1}$ is a $b_{G_{U_2}}$-Cauchy sequence tending to the boundary of G_{U_2} (in the natural topology of G_{U_2}), which, however, contradicts the Bergman completeness of G_{U_2}. $\square$

Remark 3.7. Since any Kobayashi complete bounded domain is taut (a bounded domain D in $\mathbb{C}^n$ is *taut* if for any convergent sequence of mappings $\varphi_\nu \in \mathcal{O}(E, D)$ its limit φ satisfies that $\varphi(E) \subset D$ or $\varphi(E) \subset \partial D$), there are bounded pseudoconvex balanced domains (in fact any such that the Minkowski functional is not continuous) in $\mathbb{C}^2$ such that no estimate of the type $b_D \leq Ck_D$ holds (compare [Jar-Pfl 2]).

Note that there are bounded balanced pseudoconvex domains which are not fat (i.e. $\operatorname{int}(\bar{D}) \neq D$, see [Sic]), so there are Bergman complete domains satisfying (+)

which are not fat (use Theorem 2.1). Other domains having the same property (but in the class of Hartogs domains) are given below.

Theorem 3.1, Theorem 3.5 and Theorem 3.6 apply among others to the following domain

$$G_E := \{(z,w) \in \ E \times \mathbb{C} : |w| < \exp(-\exp(\sum_{j=1}^{\infty} \alpha_j \log \frac{|z-a_j|}{2}))\},$$

where $\alpha_j > 0$, $\{a_j\}_{j=1}^{\infty}$ is dense in E_* and $\sum_{j=1}^{\infty} \alpha_j \log |a_j| > -\infty$. Note that $G_E \subset E^2$, $G_E \neq E^2$ but $\operatorname{int}(\bar{G}_E) = E^2$.

It follows from Theorem 3.6 that any bounded pseudoconvex Hartogs domain over a complete bounded pseudoconvex Reinhardt domain (e.g. the unit disc) is Bergman complete (see [Pfl 2]).

It seems to be natural to ask the question whether Theorem 3.6 remains true under the assumption that D is Bergman complete.

Since any bounded hyperconvex domain is Bergman complete, new results concerning Bergman completeness are given in non-hyperconvex case. In the class of bounded pseudoconvex balanced domains hyperconvexity is equivalent to tautness and the latter is equivalent to the continuity of the Minkowski functional associated to the domain.

Below we give a full characterization of tautness and hyperconvexity in the class of bounded pseudoconvex Hartogs domains with m-dimensional balanced fibers.

Proposition 3.8. *Let G_D be a bounded pseudoconvex Hartogs domain over D with m-dimensional balanced fibers. Then*

G_D is taut iff D is taut and H is continuous;

G_D is hyperconvex iff D is hyperconvex and H is continuous.

Proof. Note that non-continuity of H gives us the existence of the sequence $\{(z_\nu, w_\nu)\} \subset G_D$ convergent to $(z,w) \in D \times \mathbb{C}^m$ such that $\lim_{\nu\to\infty} H(z_\nu, w_\nu) = \delta < H(z,w) = 1$. Then the sequence $\varphi_\nu(\lambda) := (z^\nu, \frac{w^\nu \lambda}{H(z_\nu, w_\nu)})$, $\lambda \in E$, satisfies $\varphi_\nu(E) \subset G_D$ and φ_ν converges locally uniformly to φ, where $\varphi(\lambda) = (z, \frac{w\lambda}{\delta})$, $\varphi(0) \in G_D$ but $\varphi(E) \not\subset G_D$, so G_D cannot be taut.

It is trivial to see that tautness (respectively, hyperconvexity) of G_D implies tautness (respectively, hyperconvexity) of D.

Hyperconvexity of D delivers us the existence of negative continuous plurisubharmonic exhaustion function u of D. Note that if H is continuous, then the function $\max\{u(z), \log H(z,w)\}$ is a continuous negative exhaustion function of G_D.

Assume now tautness of D and continuity of H. Consider a sequence $\varphi^\nu := (\varphi_1^\nu, \varphi_2^\nu) \in \mathcal{O}(E, G_D)$, which converges locally uniformly to φ^0. Because of tautness of D either $\varphi_1^0 \in \mathcal{O}(E, D)$ or $\varphi_1^0(E) \subset \partial D$, in the second case $\varphi^0(E) \subset \partial G_D$. So consider the first case. It easily follows from the maximum principle for subharmonic functions that either $H(\varphi^0(\lambda)) = 1$ or $H(\varphi^0(\lambda)) < 1$, $\lambda \in E$, which finishes the proof. $\square$

4. Hartogs-Laurent domains. In this section we consider Hartogs-Laurent domains. More precisely, let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let u, v be plurisubharmonic functions on D, $u + v < 0$ on D. Then we define *the Hartogs-Laurent domain* $G := \{(z, z_{n+1}) \in D \times \mathbb{C} : \exp(u(z)) < |z_{n+1}| < \exp(-v(z))\}$ *over* D. We assume additionally that there is some constant $C \in \mathbb{R}$ such that $v(z) > C$ (i.e. G is bounded) and $u \not\equiv -\infty$.

Proposition 4.1. *Let G be as above (with some D, u and v). Assume additionally that D satisfies $(*)$. Then G satisfies $(*)$.*

Proof. Since $G \subset \{(z, z_{n+1}) \in D \times \mathbb{C} : |z_{n+1}| < \exp(-v(z))\}$, we get in view of Theorem 3.1 and because of the contraction property of the Bergman kernel under inclusion of domains that $K_G(z, z_{n+1}) \to \infty$ whenever $(z, z_{n+1}) \to (w, w_{n+1}) \in \partial G$, where $w \in \partial D$ or $|w_{n+1}| \geq \exp(-v(w))$.

Now we consider the case when $(z, z_{n+1}) \to (w, w_{n+1}) \in \partial G$, where $w \in D$ and $|w_{n+1}| \leq \exp(u(w))$.

First we prove that $K_G(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, 0) \in \partial G$ with $w \in D$. Take a small ball $U \subset\subset D$ with the centre at w. Put $G_U := G \cap (U \times \mathbb{C})$. We claim that the function $\frac{1}{z_{n+1}}$ is from $L_h^2(G_U)$. In fact,

$$\int_{G_U} \frac{1}{|z_{n+1}|^2} d\mathcal{L}^{2n+2}(z, z_{n+1}) = \int_U \Big(\int_{\exp(u(z)) < |z_{n+1}| < \exp(-v(z))} \frac{1}{|z_{n+1}|^2} d\mathcal{L}^2(z_{n+1}) \Big) d\mathcal{L}^{2n}(z) = 2\pi \int_U (-v(z) - u(z)) d\mathcal{L}^{2n}(z).$$

Therefore, in view of the local summability of plurisubharmonic functions (not identical to $-\infty$) the last expression is finite. Consequently, $K_{G_U}(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, 0)$. And now the localization property of the Bergman kernel (Theorem 1.3) implies that

$$K_G(z, z_{n+1}) \to \infty \text{ as } (z, z_{n+1}) \to (w, 0).$$

We are left with the case $(z, z_{n+1}) \to (w, w_{n+1}) \in \partial G$, $w \in D$, $0 < \varepsilon < |w_{n+1}| \leq \exp(u(w))$. Consider now the new Laurent-Hartogs domain G_1 defined over D with u replaced by $\tilde{u} := \max\{u, \log \varepsilon\}$ (and the same v). Taking now $\tilde{G}_1$ to be $\{(z, 1/z_{n+1}) : (z, z_{n+1}) \in G_1\}$ we get that the convergence of $K_{G_1}(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, w_{n+1})$ is equivalent to the convergence of $K_{\tilde{G}_1}(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, 1/w_{n+1})$ (use the invariance of the Bergman kernel with respect to biholomorphic mappings). Since $\tilde{G}_1 \subset \{(z, z_{n+1}) : z \in D,\ |z_{n+1}| < \exp(-\tilde{u}(z))\}$ and $1/|w_{n+1}| \geq \exp(-u(w))$ we get (using the contractivity of the Bergman kernel under inclusion and Theorem 3.1) that $K_{G_1}(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, w_{n+1})$. And now the localization of the Bergman kernel (Theorem 1.3) implies that $K_G(z, z_{n+1}) \to \infty$ as $(z, z_{n+1}) \to (w, w_{n+1})$. $\square$

Theorem 4.2. *Let D be a bounded domain pseudoconvex domain in $\mathbb{C}^n$, which is c^i-complete. Let G be as above with the additional property that there is some constant C such that $u(z) > C > -\infty$ for any $z \in D$. Then G is Bergman complete.*

Proof. We proceed similarly as in the proofs of results in Section 3. Take any b_G-Cauchy sequence $\{(z^\nu, z_{n+1}^\nu)\}$ converging to $(z^0, z_{n+1}^0) \in \partial G$. One easily excludes (because of c^i-completeness of D) the case $z^0 \in \partial D$. In case $z^0 \in D$ we may exactly as in the proof of Theorem 3.6 reduce the problem to the problem of completeness of $\tilde{G} := G \cap (U \times \mathbb{C})$, where U is some small ball around z^0, $U \subset\subset D$, such that $U \times A \subset G$, where A is some annulus. Similarly, as in the proof of Lemma 3.3, expanding any L^2_h function in the series $F(z, z_{n+1}) := \sum_{\nu=-\infty}^{\infty} h_\nu(z) z_{n+1}^\nu$, $(z, z_{n+1}) \in \tilde{G}$, we

easily get that $F_N(z,z_{n+1}) := \sum_{\nu=-N}^{N} h_\nu(z) z_{n+1}^\nu$ tends in $L_h^2(\tilde G)$ to F. Moreover, $h_\nu(z) \in L_h^2(U)$. Since $H^\infty(U)$ is dense in $L_h^2(U)$, we get consequently, that $H^\infty(\tilde G)$ is dense in $L_h^2(\tilde G)$ (approximate $h_\nu(z)z_{n+1}^\nu$ with $h_{\nu,j}(z)z_{n+1}^\nu$, where $h_{\nu,j} \in H^\infty(U)$ tends to h_ν in $L^2(U)$, and then use inequality $u > C$), which in connection with Proposition 4.1 and Theorem 1.4 finishes the proof. $\square$

5. Zalcman type domains. In Section 3 we saw that there are non-fat domains satisfying $(*)$ and being Bergman complete. In this chapter we go into the opposite direction and we find domains, which are bounded pseudoconvex and fat but which do not satisfy $(*)$ (which gives the answer about the existence of such domains in [Jar-Pfl 2]). This counterexample is found in the class of Zalcman type domains, which were considered in the context of $(*)$ and Bergman completeness in [Ohs 2] and [Chen]. It follows from the papers above that there are Zalcman type domains satisfying $(*)$ and being Bergman complete and non-hyperconvex. We show that there are Zalcman type domains, which neither satisfy $(*)$ nor are Bergman complete.

Let us fix a sequence (of pairwise different points) $\{a_j\}_{j=1}^\infty \subset E$ and a closed disc $B \subset E$ such that $a_j \notin B$, $a_j \to 0$ and $0 \in B$ (automatically $0 \in \partial B$).

Below we shall consider only sequence of positive numbers r_j such that $\bar\triangle(a_j,r_j) \cap \bar\triangle(a_k,r_k) = \emptyset$ for any $j \neq k$ and $B \cap \bar\triangle(a_j,r_j) = \emptyset$.

Lemma 5.1. *We may choose r_j so that there is a constant $M < \infty$ such that*

$$K_{D_N}(z) < M \text{ for any } z \in B,\ N = 1,2,\dots, \tag{5.1}$$

where $D_N := E \setminus (\bigcup_{j=1}^N \bar\triangle(a_j,r_j))$.

Proof. We define r_j inductively. Since $E \setminus \bar\triangle(a_1,r_1)$ increases to $E \setminus \{a_1\}$ as r_1 decreases to 0 and $K_{E\setminus\{a_1\}}$ coincides with K_E on $E \setminus \{a_1\}$ we get that there is a constant $M \in \mathbb{R}$ such that $K_{E\setminus\bar\triangle(a_1,r_1)} < M$ on B for sufficiently small $r_1 > 0$.

Assume that we have already chosen $r_1,\dots,r_N$ such that

$$K_{D_N} < M \text{ on } B \tag{5.2}$$

(D_N is defined as in the lemma). Since $D_N \setminus \bar\triangle(a_{N+1},r_{N+1})$ increases to $D_N \setminus \{a_{N+1}\}$ and $K_{D_N\setminus\{a_{N+1}\}}$ coincides with K_{D_N} on $D_N \setminus \{a_{N+1}\}$, we get as previously (use (5.2)) that $K_{D_N\setminus\bar\triangle(a_{N+1},r_{N+1})} < M$ on B for sufficiently small $r_{N+1} > 0$, which completes the proof. $\square$

Proposition 5.2. *There is a sequence $s_j \to 0$, $0 < s_j \le r_j$ and a domain $G := E \setminus (\bigcup_{j=1}^\infty \bar\triangle(a_j,s_j) \cup \{0\})$ satisfying the property*

$$K_G(z) < M, \quad z \in B \cap G.$$

Proof. Let us fix an increasing sequence of compact sets L_N such that $\bigcup_{N=1}^\infty L_N = \operatorname{int} B$.

We claim that we may choose a family of positive numbers $\{s_N^j\}_{1\le N<\infty, N\le j}$ such that $s_{N+1}^j \le s_N^j \le r_j$ for $j \ge N+1$ and $s_N^N \le r_N$ such that for the domain

$$G_N := E \setminus \Big(\bigcup_{j=N+1}^{\infty} \bar\triangle(a_j, s_N^j) \cup \bigcup_{j=1}^{N} \bar\triangle(a_j, s_j^j) \cup \{0\}\Big)$$

we have $K_{G_N} < M$ on L_N.

Assume for a while that such a choice can be done. Then define $s_j := s_j^j$. Since $G_N \subset G$, we have $K_{G_N} \geq K_G$ for any N, in particular, for any N $K_G < M$ on L_N, which completes the proof.

We define the desired family inductively with respect to N. Let $0 < t < 1$. Since $E \setminus (\bigcup_{j=2}^{\infty} \bar{\triangle}(a_j, tr_j) \cup \bar{\triangle}(a_1, r_1) \cup \{0\})$ increases to $E \setminus (\bigcup_{j=2}^{\infty} \{a_j\} \cup \bar{\triangle}(a_1, r_1) \cup \{0\})$ as t decreases to 0 and the Bergman kernel of the last domain is the restriction to this domain of K_{D_1}, we get that for t sufficiently small $K_{G_1} < M$ on L_1, where $s_1^1 := r_1$, $s_1^j = tr_j$, $j \geq 2$.

Assume that the construction has been succesful for N (i.e. we have defined already all s_k^j, $j \geq k$, $k \leq N$). Let $0 < t < 1$. Since $E \setminus (\bigcup_{j=N+2}^{\infty} \bar{\triangle}(a_j, ts_N^j) \cup \bar{\triangle}(a_{N+1}, s_N^{N+1}) \cup \bigcup_{j=1}^{N} \bar{\triangle}(a_j, s_j^j) \cup \{0\})$ increases to $E \setminus (\bigcup_{j=N+2}^{\infty} \{a_j\} \cup \bar{\triangle}(a_{N+1}, s_N^{N+1}) \cup \bigcup_{j=1}^{N} \bar{\triangle}(a_j, s_j^j) \cup \{0\})$ as t decreases to 0 and the Bergman kernel of the last domain is the restriction to this domain of $K_{E \setminus (\bar{\triangle}(a_{N+1}, s_N^{N+1}) \cup \bigcup_{j=1}^{N} \bar{\triangle}(a_j, s_j^j))}$ (which is smaller than or equal to $K_{D_{N+1}}$ because D_{N+1} is a subset of the considered domain), we get that defining for t sufficiently small $s_{N+1}^{N+1} := s_N^{N+1}$, $s_{N+1}^j := ts_N^j$, $j \geq N+2$ the inequality $K_{G_{N+1}} < M$ holds on L_{N+1}. $\square$

Let us remark that because of the property $K_{D_j}(z) \to K_D(z)$ locally uniformly for any sequence $\{D_j\}_{j=1}^{\infty}$ of domains such that $D_j \subset D_{j+1}$ and $\bigcup_{j=1}^{\infty} D_j = D$ (D is a bounded domain) we get easily that $\beta_{D_j} \to \beta_D$ locally uniformly on $D \times \mathbb{C}^n$ (although the convergence in contrast to the convergence of Bergman kernels need not be monoton).

Based on the above property of the Bergman kernel we present below a similar construction (to that from Proposition 5.2) leading to a domain having the assumptions as in Proposition 5.2 and, additionally, not Bergman complete. We denote $\beta_D(z) := \beta_D(z; 1)$.

Lemma 5.3. *There are a constant $M_1 \in \mathbb{R}$ and a family of tuples $\Lambda = \bigcup_{N=0}^{\infty} \Lambda_N$, where:*

$$\Lambda_0 = \emptyset,\ \Lambda_N \subset (0, s_1] \times \ldots \times (0, s_N];$$

for any N if $\lambda \in \Lambda_N$ then there is some λ_{N+1} such that for any $0 < s \leq \lambda_{N+1}$ $(\lambda, s) \in \Lambda_{N+1}$;
for any $\lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda$ we have that $\beta_{D_\lambda} < M_1$ on B, where

$$D_\lambda := E \setminus \bigcup_{j=1}^{N} \bar{\triangle}(a_j, \lambda_j).$$

Proof. The proof goes similarily as that of Lemma 5.1. We proceed using induction. Since $E \setminus \bar{\triangle}(a_1, t)$ increases to $E \setminus \{a_1\}$ as t decreases to 0 and $\beta_{E \setminus \{a_1\}}$ coincides with β_E on $E \setminus \{a_1\}$ we get that there is a constant $M_1 \in \mathbb{R}$ such that $\beta_{E \setminus \bar{\triangle}(a_1, t)} < M_1$ on B for any $0 < t \leq \lambda_1 \leq s_1$. We define $\Lambda_1 := (0, \lambda_1]$.

Assume that we have already defined $\Lambda_1, \ldots, \Lambda_N$ such that Lemma is satisfied, in particular,

$$\beta_{D_\lambda} < M_1 \text{ on } B$$

for any $\lambda \in \Lambda_N$

Fix any $\lambda \in \Lambda_N$. Since $D_\lambda \setminus \bar{\triangle}(a_{N+1}, t)$ increases to $D_\lambda \setminus \{a_{N+1}\}$ and $\beta_{D_\lambda \setminus \{a_{N+1}\}}$ coincides with β_{D_λ} on $D_\lambda \setminus \{a_{N+1}\}$, we get as previously that $\beta_{D_\lambda \setminus \bar{\triangle}(a_{N+1},t)} < M_1$ on B for sufficiently small $t > 0$, which completes the proof. $\square$

Proposition 5.4. *There is a sequence $\lambda_j \to 0$, $0 < \lambda_j \le s_j$ and a domain $G := E \setminus (\bigcup_{j=1}^\infty \bar{\triangle}(a_j, \lambda_j) \cup \{0\})$ satisfying the property*

$$\beta_G(z) \le M_1, \quad z \in B \cap G.$$

Proof. Let us fix an increasing sequence of compact sets L_N such that $\bigcup_{N=1}^\infty L_N = \operatorname{int} B$.

Without loss of generality we may assume that $s_1 = \lambda^1$ (s_1 is from Proposition 5.2 and λ^1 from Lemma 5.3).

It is sufficient to find sequences $\{\lambda^j\}_{j=1}^\infty$ and $\{t_j\}_{j=1}^\infty \subset (0,1)^{\mathbb{N}}$ such that $\lambda^N \in \Lambda_N$, $\lambda_N = t_1 \cdot \ldots \cdot t_N s_N$, $\lambda^{N+1} = (\lambda^N, \lambda_{N+1})$, and $\beta_{D_N} < M_1$ on L_N, where $D_N := D_{\lambda^N} \setminus \bigcup_{j=N+1}^\infty \bar{\triangle}(a_j, t_1 \cdot \ldots \cdot t_N s_j)$.

Put $\lambda_1 := \lambda^1 (= s_1)$, $t_1 := 1$. Then for $1 > t > 0$ small enough the Bergman metric on $D_{\lambda^1} \setminus (\bigcup_{j=2}^\infty \bar{\triangle}(a_j, ts_j) \cup \{0\})$ is less than M_1 on L^1 for $0 < t \le t_2 < 1$, we may also assume that $\lambda^2 := (\lambda^1, t_2 s_2) \in \Lambda^2$.

Assume that the construction has been succesful for N (i.e. we have defined already all t_j, $j = 1, \ldots, N$ and λ^j, $j = 1, \ldots, N$). Let $0 < t < 1$. Since $D_{\lambda^N} \setminus (\bigcup_{j=N+1}^\infty \bar{\triangle}(a_j, tt_1 \cdot \ldots \cdot t_N s_j) \cup \{0\})$ increases to $D_{\lambda^N} \setminus (\bigcup_{j=N+1}^\infty \{a_j\} \cup \{0\})$ as t decreases to 0 and the Bergman metric of the last domain is the restriction to this domain of $\beta_{D_{\lambda^N}}$ we may choose t_{N+1} and then define $\lambda^{N+1} := (\lambda^N, t_1 \cdot \ldots \cdot t_{N+1} s_{N+1}) \in \Lambda^{N+1}$ having the desired properties. $\square$

Remark 5.5. Note that the above mentioned results may be put in some more general context. Two principal properties that were used were the following both: K_D and β_D do not change after deleting a discrete subset and both are continuous with respect to the increasing family of domains. Applying the same procedure we may prove for instance that there are Zalcman type domains, which are not Carathéodory complete. Consequently, there are Zalcman type domains, without peak functions in 0 (see [Rud]).

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